Model Validation Challenge Problem: Static Frame Problem.

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1   Prediction problem

Consider the structure shown on Figure 1, which is made of four bars connected by perfect hinges. The rods 1, 2 and 3 are in tension while the beam number 4 is in compression and also subject to bending by a uniform load.

The regulatory assesment problem is to specify what is the probability that the vertical displacement at the midpoint of beam 4 will not exceed a given limit.

The material of the bars and beam is randomly homogeneous, and may be described as a homogeneous random field. In addition, the stress-strain relation is in the linear range. The modulus of elasticity and its probabilistic properties are not known and have to be characterized using available data. The data available to the analyst come from experiments, namely calibration, validation and accreditation experiments, which are described below. The experiments are related to two basic pieces of information: a) the material is random and heterogeneous and b) the type of regulatory assessment based on a given quantity of interest. Besides, for the sake of the exercise, there are three different cases characterized by three different sets of data, which are distinguished by the amount of experiments. These three cases have to be analyzed separately to see the effect of the number of experiments on the conclusions of the analyst, which is the major point of the exercise.

For the analysis of the data, a mathematical model based on linear elasticity, consisting of one dimensional tension for the rods and a one dimensional Bernoulli beam for the bending of the beam, should be assumed. Please observe that this assumption implies that the Young modulus is constant over the crossections of the bars and beam under study. Besides, the influence of the compression in the bending model can be neglected.

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It is assumed that the geometry and the loads for all the cases are perfectly known, and that there are no measurement errors in all the available data.

It is assumed that the same material is used to build the structures in all experiments and in the intended prediction. However, it is also assumed that each rod and each beam are coming from sources that have the same probability distribution but are independent. The core of the excercise is the probabilistic characterization of an unknown heterogeneous material by its modulus of elasticity (which is itself a random function).

Remark 1 (Virtual experiments) *The material properties mentioned in the tables are not related to a particular material, they have been produced by virtual experiments with an artificial model which yields a modulus of elasticity that varies within the length of the bars.*

1.1 Geometry of the structure

Figure 1 shows the structure and its load. The bent bar number 4 is loaded by a uniform load of intensity $q = 6 (KN/m)$. The geometric characteristics of the bars are given in Table 1.
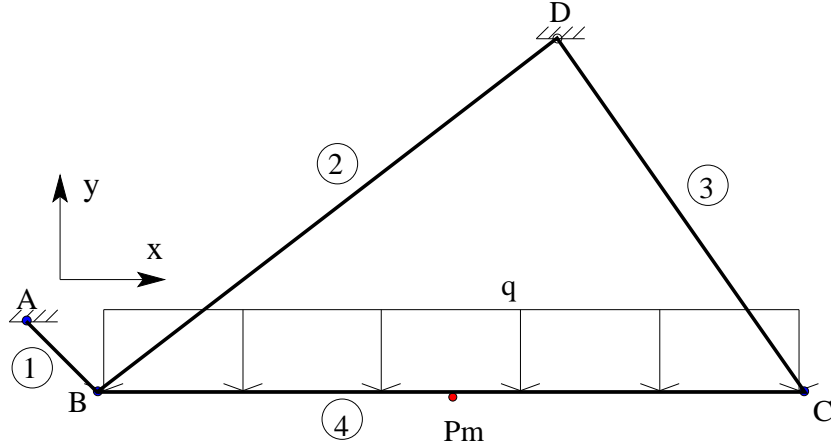

Figure 1: Prediction case. Structure and uniform load q under study. We are interested in the vertical displacement of point P_m. This displacement should be computed following Appendix A.

All hinges are assumed perfect (i.e. there are no moments at the end of the bars) and the hinges supports at points A and D are completely rigid.

Denoting by $w(P_m)$ the vertical displacement of the midpoint of beam 4, P_m, the regulatory assessment is the probability of the event

$$|w(P_m)| \leq 3.0 (mm).$$

Point	x(cm)	y(cm)	Bar #	$A(cm^2)$	$I(cm^4)$
A	0	20	1	16	
B	20	0	2	16	
C	220	0	3	16	
D	150	100	4	80	5333

Table 1: Geometry of the bars and beam from Figure 1. Observe that only beam 4 is subject to bending.

The structure is statically determined and hence the tensile stresses in all the bars and the bending moment in bar 4 are independent of the material properties. The corresponding values are given on Table 2.

Bar #	Tensile Force (KN)	Moment (KNm)
1	2.214	
2	7.274	
3	7.324	
4	-4.200	$\frac{q}{2}(x_C - x)(x - x_B)$

Table 2: Prediction case. Tensile stress of the bars from Figure 1. The value of x should be in meters for the computation of the bending moment.

The material of the bars is linear and the displacements are sufficiently small so that the classical strength of materials approach can be used. Besides, the combined effect of compression and bending should be neglected for bar number 4.

Hence, the value of the vertical displacement $w(q)$ is given by the displacements of the hinges B and C and the displacement of the simply supported beam 4. It is assumed that the material that makes the different bars is made of the same material but come from independent sources.

2   Material Properties –Calibration

The calibration experiment for the determination of the material property consists on the classical dog bone experiment. For the sake of simplicity, we will assume that the sample has a constant cross section and that the grasping occurs at the end points only.

For each sample, the measured values are the strain in the midpoint R (by a strain gage) and the total elongation of the sample under a force $F = 1.2(KNewtons) \approx 122(Kgf)$. The characteristics of the sample bars are $A = 4.0(cm^2)$ and $L = 20(cm)$, respectively.

A schematic representation of this experiment is shown on Figure 2. The measured data are given on Table 6 where the horizontal lines indicate different
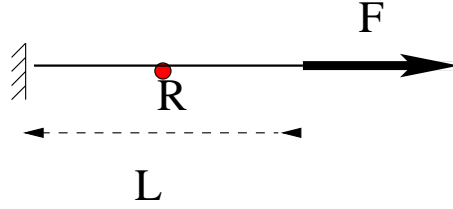
Figure 2: Calibration scheme. The available measurements from this experiment are the elongation from the initial length L (denoted by δL on Table 6) and the strain at the midpoint R. The latter value can be directly related to a local measurement of the modulus of elasticity at the midpoint (see the third column of Table 6).

data sets, see table 5.

3   Validation

The validation experiments consist of longer bars subject to tension, similarly as in the calibration. The main difference is that only the total elongation is measured. Here the characteristics of the sample bars are their crossection area $A = 4(cm^2)$ and their length, $L_v = 80(cm)$, respectively. The tensile force is again $F = 1.2(KNewtons) \approx 122(Kgf)$.

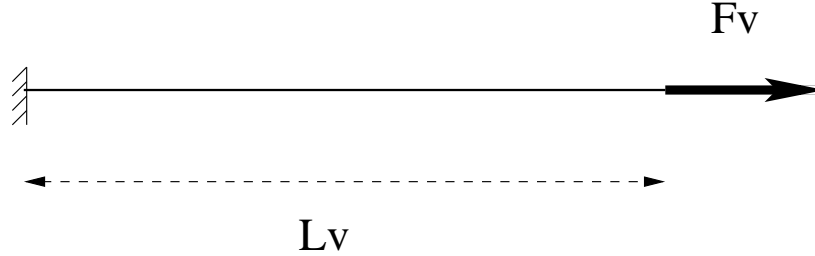A schematic representation of this experiment is shown on Figure 3. The

Figure 3: Validation scheme. The available measurement from this experiment is the elongation from the initial length L_v (denoted by δL on Table 7).

measured data are given on Table 7.

4   Accreditation

The Accreditation experiment consists of a structure subject to a point load of strength $P = 6(KN)$ at the midpoint of bar number 1 , Q, as shown on Figure 4. The measured value in this experiment is the vertical displacement of the midpoint in bar number 1.
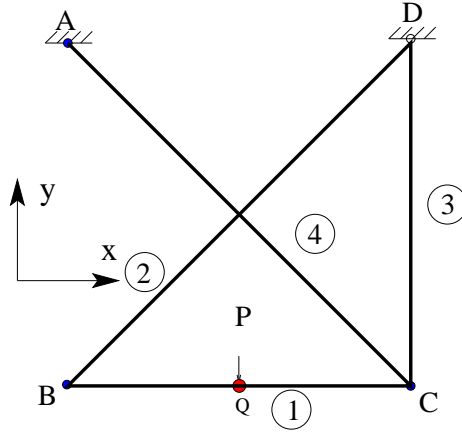
Figure 4: Set up geometry for the accreditation experiment. The available measurement from this experiment is the vertical displacement of the midpoint of beam 1 (denoted here as Q) due to a vertical point load P that is applied to the structure at Q. See Appendix A for useful formulae.

The geometric characteristics of the bars are given in Table 3. Please note that bars 2 and 4 are not connected.

Point	x(cm)	y(cm)	Bar #	$A(cm^2)$	$I(cm^4)$
A	0	50	1	16	333.3
B	0	0	2	16	
C	50	0	3	16	
D	50	50	4	20	

Table 3: Geometry of the bars and beam from Figure 4. Only beam 1 is subject to bending.

The structure is statically determined and hence the tensile stresses in all the bars and the bending moment in beam 1 are independent of the material properties. The corresponding values are given on Table 4. Observe that both cases 1 and 2 provide a single measurement for the accreditation experiment and that only case 3 has two of such measurements available.

As said before, the material of the bars is assumed linear and the displacements are sufficiently small so that the classical strength of materials approach can be used. Besides, the combined effect of compression and bending should be neglected for bar number 1.

Hence, the value of the vertical displacement $w(Q)$ is given by the displacements of the hinges B and C and the displacement of the simply supported beam 1.

The measured data are given on Table 8.

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Bar #	Tensile Force (KN)	Moment (KNm)
1	-3.000	$\frac{P}{2}\left((x - x_B)\, 1_{\{x-x_B<L_1/2\}} + (x_C - x)\, 1_{\{L_1/2<x-x_B\}}\right)$
2	4.243	
3	0.000	
4	4.243	

Table 4: Accreditation case. Tensile stress of the bars from Figure 4. The value of x should be in meters for the computation of the bending moment.

5   Regulatory Assessment

In order to assess the confidence in the regulatory assessment based on the model described in Appendix A and the limited data it is necessary to determine what can be said about the probability that the displacement in the midpoint of the beam 4, see Figure 1, will not exceed 3 milimeters using each of the different experimental data cases.

There are three cases to consider for the analysis, each of them corresponding to increasing numbers of calibration, validation and accreditation experiments, N_c, N_v and N_a, respectively. To this end, the analyst should use *the first N_c, N_v and N_a* experiments from Tables 6, 7 and 8, respectively. The different cases are defined on table 5. For instance, in the first case there are only 5 Calibration experiments and these numbers are the first 5 shown on Table 6. In the second case, there are only 20 Calibration experiments and these numbers are the first 20 shown on Table 6. Similarly, in case 3 all 30 data are used. Analogous procedure should be followed with Validation and Accreditation experimental data.

	N_c	N_v	N_a
Case 1	5	2	1
Case 2	20	4	1
Case 3	30	10	2

Table 5: Number of Calibration, Validation and Accreditation experiments for different cases.

Sample #	δL(mm)	$E(L_c/2)$ (GPa)
1	5.15e-02	13.26
2	5.35e-02	10.86
3	5.24e-02	14.77
4	5.51e-02	10.94
5	5.14e-02	11.05
6	5.38e-02	11.06
7	4.97e-02	11.97
8	5.41e-02	11.66
9	4.95e-02	12.09
10	5.42e-02	11.30
11	5.47e-02	10.98
12	5.74e-02	11.92
13	5.36e-02	11.12
14	5.42e-02	12.00
15	5.34e-02	10.98
16	5.60e-02	10.71
17	5.06e-02	10.91
18	4.99e-02	11.89
19	5.22e-02	11.43
20	5.57e-02	10.87
21	5.28e-02	11.75
22	5.10e-02	13.47
23	5.48e-02	11.44
24	5.35e-02	12.44
25	4.92e-02	12.13
26	5.51e-02	11.38
27	5.27e-02	10.75
28	5.14e-02	11.92
29	5.61e-02	10.82
30	5.56e-02	11.04

Table 6: Results of the Calibration experiments

Sample #	δL(mm)
1	2.01e-01
2	2.06e-01
3	2.01e-01
4	2.08e-01
5	2.04e-01
6	2.01e-01
7	2.06e-01
8	2.11e-01
9	1.98e-01
10	2.08e-01

Table 7: Results of the Validation experiments corresponding to the cases defined on Table 5.

Sample #	$w(P)$(mm)
1	-6.50e-01
2	-6.73e-01

Table 8: Results of the Accreditation experiments. The first measurement is used for cases 1 and 2. All measurements are available in case 3, cf. Table 5.

A   Useful formulae

A.1   Relation between Force and Elongation in rods.

The elongation on each rod of length L satisfies

$$\delta L = \frac{F}{A} \int_0^L \frac{ds}{E(s)} \tag{1}$$

where

- A is the (constant) cross sectional area,

- F is the applied force, which is positive if the rod is in tension and negative in compression,

- $E(s)$ is the value of the modulus of elasticity at position s, $0 \leq s \leq L$.

A.2   Relation between hinges displacements and rod elongations

A.2.1   Accreditation

$$\begin{pmatrix} \delta L_1 \\ \delta L_2 \\ \delta L_3 \\ \delta L_4 \end{pmatrix} = \begin{pmatrix} -1 & 1 & 0 & 0 \\ -1/\sqrt{2} & 0 & -1/\sqrt{2} & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 1/\sqrt{2} & 0 & -1/\sqrt{2} \end{pmatrix} \begin{pmatrix} \delta x_B \\ \delta x_C \\ \delta y_B \\ \delta y_C \end{pmatrix} \tag{2}$$

A.2.2   Prediction

$$\begin{pmatrix} \delta L_1 \\ \delta L_2 \\ \delta L_3 \\ \delta L_4 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} & 0 \\ -0.7926 & 0 & -0.6097 & 0 \\ 0 & 0.5735 & 0 & -0.8192 \\ -1 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \delta x_B \\ \delta x_C \\ \delta y_B \\ \delta y_C \end{pmatrix} \tag{3}$$

A.3   Displacement Computations

For each of the cases (Calibration, Validation, Accreditation and Prediction) the procedure for computing the desired displacements is the following:

STEP 1. Find the forces acting on each rod. This is possible because all proposed cases are statically determined and the resulting forces are tabulated (see Table 4 for Accreditation and Table 2 for Prediction where the mentioned forces are displayed).

STEP 2. By means of relation (1) find the elongations on each rod using the forces found in [**STEP 1.**].

STEP 3. For the Accreditation and Prediction cases, use the relations between hinges displacements and rod elongations, equations (2) and (3) respectively, to determine the hinges displacements from the elongations found in [**STEP 2.**]

STEP 4. Let the auxiliary function

$$\varphi(s) = \begin{cases} s/2, & \text{if } 0 \leq s < L/2 \\ (L-s)/2, & \text{if } L/2 \leq s \leq L. \end{cases}$$

Here L is the length of the beam under consideration.

For the Accreditation and Prediction cases, use the following formulae to find the vertical displacement of the midpoint of the beam:

Accreditation Case

$$w(Q) = (\delta y_B + \delta y_C)/2 - \frac{1}{I} \int_0^{L_1} \frac{M(s)}{E(s)} \varphi(s) ds$$

where
$$M(s) = P\varphi(s)$$

and I is the constant crossectional moment of inertia of the beam 1.

Prediction Case

$$w(P_m) = (\delta y_B + \delta y_C)/2 - \frac{1}{I} \int_0^{L_4} \frac{M(s)}{E(s)} \varphi(s) ds$$

where
$$M(s) = \frac{q}{2}(L_4 - s)s.$$

and I is the constant crossectional moment of inertia of the beam 4.